DATE OF EXAM **Solution**
SUBJECT NAME: Advanced Functional Analysis - FINAL OR MIDTERM Exam: Mid-Semester 2016 - Semester ENTER I or II: M. Math IInd Yr

1. *Let X be a complex normed linear space. Let $f : X \to C$ be a non-zero linear map. Show that either $f(\{x \in X : ||x|| \leq 1\})$ is a bounded set or is all of C. In the second case show that $ker(f)$ is dense in X.*

   Solution: Since $f : X \to C$ is non-zero, there exists $x \in X$ such that $f(x) = r \neq 0$. Since f is linear, $f(\frac{x}{r}) = 1$, take $y = \frac{x}{r}$, therefore $f(y) = 1$. For each $\lambda \in C$, we have $f(\lambda y) = \lambda f(y) = \lambda$. Therefore f maps all of C and f is not bounded map.

   Now we prove that $ker(f)$ is dense in X. We know that $f(\{x \in X : ||x|| \leq 1\})$ is not bounded. We can find a sequence of non-zero vectors $x_n \in \{x \in X : ||x|| \leq 1\}$ with $|f(x_n)| \geq n$ for each $n = 1, 2, 3, ...$ Now suppose $x \notin ker(f)$, consider the sequence $z_n = x - \frac{f(x)}{f(x_n)} x_n$. We can easily see that $f(z_n) = 0$ therefore $z_n \in ker(f)$ for each $n = 1, 2, 3, ...$ We have $||z_n - x|| = ||\frac{f(x)}{f(x_n)} x_n|| \leq ||\frac{f(x)}{f(x_n)}|| \xrightarrow{n\to\infty} 0$. Therefore it follows that $x \in \overline{ker(f)}$. As x was an arbitrary element not in $ker(f)$. Hence $\overline{ker(f)} = X$. □

2. *Show that for any Banach space X,*

$$\bigoplus_{\infty} X = \{\{x_n\}_{n\geq 1} : x_n \in X, \ \sup_{n\geq 1} ||x_n|| < \infty\}$$

   equipped with the norm $||\{x_n\}_{n\geq 1}|| = \sup_{n\geq 1} ||x_n||$ is a Banach space.

   Solution: Let $x^1, x^2, ...$ be a Cauchy sequence in $\bigoplus_{\infty} X$. Here $x^k = (x_1^k, x_2^k, ...)$ for each $k = 1, 2, ...$ We have to find an element x in $\bigoplus_{\infty} X$ such that x^k converges to x. Given $\epsilon > 0$, since x^k is a Cauchy sequence, we see that there exists an $N > 0$ such that $||x^k - x^l||_\infty < \epsilon$ for all $k, l > N$. Thus $\sup_{n\geq 1} ||x_n^k - x_n^l|| < \epsilon$ for all $k, l > N$. In particular, we have $||x_n^k - x_n^l|| < \epsilon$ for all n and for all $k, l > N$. This means that for each n, the sequence $x_n^1, x_n^2, ...$ is a Cauchy sequence in X. Since X is complete, thus we have a limit call it x_n, therefore $\lim_{k\to\infty} x_n^k = x_n$. Let x denote the sequence $x = (x_1, x_2, ...)$ and $x \in \bigoplus_{\infty} X$. We show that x^k converges to x. Choose $\frac{\epsilon}{2}$, we can find an $N > 0$ such that $||x_n^k - x_n^l|| \leq \frac{\epsilon}{2}$ for all n and for all $k, l > N$. Taking limit $l \to \infty$, we obtain $||x_n^k - x_n|| \leq \frac{\epsilon}{2}$ for all n and for all $k, l > N$. $\sup_{n\geq 1} ||x_n^k - x_n|| \leq \frac{\epsilon}{2}$ for all n and for all $k > N$. There for $||x^k - x||_\infty \leq \frac{\epsilon}{2} < \epsilon$ for all $k > N$. This implies that x^k converges to x. □

3. *Let $M = \{f \in C([0, 1]) : f([0, \frac{1}{2}]) = 0\}$. On the quotient space, let $\Phi : C([0, 1])/M \to C([0, \frac{1}{2}])$ be defined by $\Phi([f]) = f|_{[0,\frac{1}{2}]}$. Show that Φ is a well-defind, linear, onto, isometry.*

   Solution: First we show that Φ is well-defined. Let $[f], [g] \in C([0, 1])/M$, if $[f] = [g]$ then $f(x) = g(x), \forall x \in [0, \frac{1}{2}]$, $f|_{[0,\frac{1}{2}]} = g|_{[0,\frac{1}{2}]}$, thus $\Phi([f]) = \Phi([g])$, there fore Φ is well-defined.

   Now we show that Φ is linear. Let $[f], [g] \in C([0, 1])/M$, $\Phi([f] + [g]) = \Phi([f + g]) = f + g|_{[0,\frac{1}{2}]} = f|_{[0,\frac{1}{2}]} + g|_{[0,\frac{1}{2}]} = \Phi([f]) + \Phi([g])$. Let $[f] \in C([0, 1])/M$ and c is a scalar. $\Phi(c[f]) = \Phi([cf]) = (cf)|_{[0,\frac{1}{2}]} = cf|_{[0,\frac{1}{2}]} = c\Phi([f])$.

Now we show that Φ is onto. Let $g \in C([0, \frac{1}{2}])$. Define a map $f : [0, 1] \to [0, 1]$ by $f(x) := g(x)$ if $x \in [0, \frac{1}{2}]$ and $f(x) = g(\frac{1}{2})$ for $x \in [\frac{1}{2}, 1]$. Then, clearly $f \in C([0, 1])$ and $\Phi([f]) = f|_{[0,\frac{1}{2}]} = g$.

Now we show that Φ is an isometry. Let the norm on $C([0, 1])/M$ is denoted by $|||\cdot|||$ and is defined by $|||f + M||| := \inf\{\|f + g\| : g \in M\}$ for any $f \in C([0, 1])$. Now, $|||[f]||| = \inf\{\|f + g\| : g \in M\}$. Let $\lambda := \|f|_{[0,\frac{1}{2}]}\| = \sup\{|f(x)| : x \in [0, \frac{1}{2}]\}$ and $S := \{\|f + g\| : f \in C([0, 1]), g \in M\}$. For $g \in M$ and $f \in C([0, 1])$, $\|f + g\| = \sup\{|f(x) + g(x)| : x \in [0, 1]\} = \text{Max}\{\{\sup\{|f(x) + g(x)| : x \in [0, \frac{1}{2}]\}, \sup\{|f(x)+g(x)| : x \in [\frac{1}{2}, 1]\}\} = \max\{\lambda, \sup\{|f(x)+g(x)| : x \in [\frac{1}{2}, 1]\}\} \geq \lambda$ i.e. λ is a lower bound of S. Now, let $\sup\{|f(x) + g(x)| : x \in [0, \frac{1}{2}] = c_1$ and $\sup\{|f(x) + g(x)| : x \in [\frac{1}{2}, 1]\} = c_2$. If $c_1 \geq c_2$ then we choose $g = 0$ and get $\inf(S) = \lambda$. If $c_1 < c_2$ then assume that f attains c_2 at some points. Let p be a point in $[\frac{1}{2}, 1]$ such that $f(p) = c_2$ and $f(x) < c_2$ for $x < p$. Now, construct a function $g \in M$ such that g is a line joining the poins $(\frac{1}{2}, 0)$ and $(0, -c_2)$ and $g(x) = -c_2$ for $x > p$. Then $\|f + g\| = \lambda$. And hence, $\lambda = \inf(S) = |||[f]|||$.

$\square$

4. *Let X, Y be a LCTVS spaces. Let $T : X \to Y$ be an isomorphism. Suppose X^* and Y^* are equipped with the weak*-topology. Show that $T^* : Y^* \to X^*$ is an isomorphism.*

   Solution: Let $T^* : Y^* \to X^*$ is defined by $T^*(y^*)(x) = y^*(T(x))$.
   We show that T^* is linear, let $y_1^*, y_2^* \in Y^*$, $T^*(y_1^* + y_2^*)(x) = T^*((y_1 + y_2)^*)(x) = (y_1 + y_2)^*(T(x)) = (y_1^* + y_2^*)(T(x)) = y_1^*(T(x)) + y_2^*(T(x)) = T^*(y_1^*)(x) + T^*(y_2^*)(x)$. Let c be a scalar and $y^* \in Y^*$, $T^*(cy^*)(x) = T^*((cy)^*)(x) = (cy)^*(T(x)) = cy^*(T(x)) = cT^*(cy^*)$.
   We show that T^* is one to one and onto, $T^*(y_1^*)(x) = T^*(y_2^*)(x)$, $y_1^*(T(x)) = y_2^*(T(x))\ \forall x$, thus $y_1^* = y_2^*$. Therefore T^* is one to one. For $x^* \in X^*$, take $y^* = x^* \circ T^{-1}$, then $T^*(y^*) = x^*$, hence T is onto.
   We now show that T^* and $(T^*)^{-1}$ are continuous. $T^*(y^*) = y^* \circ T$, Since T is continuous therefore T^* is continuous. We know $(T^*)^{-1} = (T^{-1})^*$, the continuity of T^{-1} will imply that the continuity of $(T^*)^{-1}$. $\square$

5. *Give examples of two normed linear spaces, and a continuous linear map T between them such that T^* has closed range but the range of T is not closed.*

   Solution: Let c_{00} be space of all sequences which have only finitely many nonzero elements with sup norm and Let c_0 be space of all sequences converges to 0 with sup norm. Take the identity map $I : c_{00} \to c_0$, the range of I is not closed since c_{00} is dense subspace of c_0. But $c_0^* = c_{00}^* = l^1$, I^* is the identity maps from l^1 to l^1, therefore it is easy to see that I^* has closed range. $\square$

6. *Construct a sequence $f_n : [0, 1] \to [0, 1]$ of continuous functions such that $||f_n|| = 1$ for all $n \geq 1$, $f_n(t) \to 0$ for all $t \in [0, 1]$. Use the Riesz representation theorem to show that $f_n \to 0$ in the weak topology of $C[0, 1]$.*

   Solution: We define $f_n(t)$ as follows,

$$f_n(t) = \begin{cases} nt & \text{if } 0 \leq t \leq \frac{1}{n} \\ 2 - nt & \text{if } \frac{1}{n} < t \leq \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} < t \leq 1 \end{cases}$$

   then $f_n(t) \in C[0, 1]$, $||f_n|| = 1$ for $n = 1, 2, ...$ and $f_n(t) \to 0$ for all $t \in [0, 1]$.

Now we prove that $f_n \to 0$ in the weak topology of $C[0,1]$. We have $||f_n|| = 1$ for $n = 1, 2, ...$ and $f_n(t) \to 0$ for all $t \in [0,1]$. If $f^* \in (C[0,1])^*$, then by Riesz representation theorem there exists a measure μ such that $f^*(f) = \int\limits_0^1 f d\mu$ for every $f \in C[0,1]$.

We have $|f^*(f_n) - f^*(0)| = |\int\limits_0^1 (f_n(t) - 0) d\mu| \leq \int\limits_0^1 |f_n(t) - 0| d\mu$.

Now, $f_n(t) \to 0$ for every $t \in [0,1]$ and $|f_n(t)| = 1$. Hence by dominated convergence theorem, we see that $f^*(f_n) \to 0$. Thus $f_n \to 0$ in the weak topology.

$\square$

7. *Let $D = \{z : |z| < 1\}$ be the open unit disk. Let $A(D)$ denote the space of analytic functions on D with family of semi-norms, $p_z(a) = |a(z)|$ for $z \in D$ and $a \in D$. Let $F = \{p \in A(D) : p$ is a polynomial of degree at most $n\}$. Show that $A(D) = F \bigoplus Y$ (direct sum) for some closed subspace $Y \subset A(D)$.*

   Solution: Since $A(D)$ is locally convex topological vector space and dim $F = n+1 < \infty$. Using the Lemma 4.21(a), page-106 from the book Walter Rudin, Functional Analysis, F is complemented in $A(D)$. Therefore there exist a closed subspace $Y \subset A(D)$ such that $A(D) = F \bigoplus Y$. $\square$

8. *Let X be a LCTVS space. Let $C \subset X$ be a closed convex set. Show that C is also closed in the weak-topology.*

   Solution: Fix $y \in C^c$, Using Hahn-Banach separation theorem, there exists a linear functional ϕ such that $\phi(y) < \inf_{x \in C} \phi(x)$. This ϕ provides a weak open set containing y and disjoint from C. Thus $X - C$ is weakly open. Hence C is weakly closed.

   $\square$